A NEW STATISTICAL DESCRIPTION
OF HADRONIC AND e+e− MULTIPLICITY DISTRIBUTIONS

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A new statistical description of hadronic charged-particle multiplicity distributions is presented, which takes into account the associated neutral multiplicity and charge conservation. A simple gamma distribution is used to generate fictitious intermediate neutral objects that have equal probability of becoming single neutral hadrons or charged hadron pairs. The resulting charged multiplicity distributions are very similar to those generated by the negative binomial distribution, which has been very successful in describing the data. Diffractive, non-diffractive and single-jet multiplicities are well fitted by this procedure. Good fits to e+e− charged multiplicity distributions are also obtained by assuming that each such distribution is the result of the statistical addition of two identical and independent single-jet hadronic-type distributions. The success in fitting e+e− data by using hadronic single-jet distributions provides a new insight on universality.

1. Introduction. The concept of scaling of charged-particle multiplicity distributions, brought into focus for the case of hadron–hadron collisions by Koba, Nielsen and Olesen in 1972 [1] and hence known as KNO scaling, has spurred an enormous amount of experimental and theoretical activity, which is presently continuing. KNO scaling, derived on the assumption of the validity of Feynman scaling [2], states that the probability P_n for n charged particles in the final state multiplied by the mean multiplicity $\langle n \rangle$ is only a function of z, $\langle n \rangle P_n = \psi(z)$, where $z = n/\langle n \rangle$ is the scaled multiplicity. There is no energy dependence in this formula other than that implied by the relation between $\langle n \rangle$ and $\sqrt{s}$. The scaled multiplicity moments are energy-independent constants. This property restricts the form of statistical functions that can be used to describe multiplicity distributions that obey KNO scaling. For example, the Poisson distribution, for which the scaled width is given by $D/\langle n \rangle = \langle n \rangle^{-1/2}$ is excluded, since $\langle n \rangle$ is a function of $\sqrt{s}$.

A well known statistical function that can be adopted to obeying KNO scaling is the gaussian distribution, given in terms of scaled variables by

$$\langle n \rangle P_n^{\text{Gauss}} = (k/2\pi)^{1/2} \exp[\tfrac{1}{2}k(z-1)^2] , \qquad (1)$$

$$k^{-1} = (D/\langle n \rangle)^2 . \qquad \text{(1 cont'd)}$$

Clearly, for constant k this is a function of z only. A gaussian function with $\langle n \rangle/D = 2$ ($k=4$) and $\langle n \rangle = 2\sqrt{M}$, where M is the "available" mass in GeV, gave excellent fits to diffractive charged multiplicities [3]. In performing these fits the part of the gaussian that extends to negative values of z was ignored. Naturally, the remaining "truncated gaussian" (TG) has a slightly larger mean value $\langle n \rangle$ and a smaller width D than the input values, in agreement with the data.

The TG function was then applied [4] to inclusive pp and p̄p data, assuming that the non-diffractive (ND) component behaves exactly as the diffractive, i.e. using as input $\langle n \rangle/D = 2$ and $\langle n \rangle = 2\sqrt{M}$ with $M = \sqrt{s}$. The fits obtained in this manner were excellent. The variation of $\langle n \rangle/D$ with energy observed in the data was explained as being due to edge effects (charges of initial particles, available mass) and/or due to the superposition of the diffractive (low-multiplicity) and the ND (high-multiplicity) components. This analysis was the first one to show that:

(a) There is universality between diffractive and ND charged multiplicity distributions.

(b) When diffractive and ND multiplicities are

considered independently, KNO scaling holds in the energy range from a few GeV to ~ 60 GeV.

These conclusions were confirmed later by the direct measurement of ND multiplicities at the ISR [5]. The conclusion on universality was subsequently strengthened by results on diffractive data in the range $20 < M < 140$ GeV, obtained at the S$\bar{\text{p}}$pS collider by the UA4 collaboration [6]; it was shown that the average multiplicity $\langle n \rangle$ and the central pseudorapidity density $dn/d\eta|_{\eta=0}$ at a given diffractive mass M are the same as those in ND data at $\sqrt{s} = M$.

In spite of its success in describing well all data below $\sqrt{s} \sim 60$ GeV, in uncovering universality and in demonstrating that when diffractive and ND processes are considered independently KNO scaling works down to energies of a few GeV, the TG was merely a phenomenological tool useful in relating data. It could never be considered seriously as a candidate for a function which could provide the correct theoretical description of multiparticle production, since the necessity to truncate it lacks physical meaning. Furthermore, when the results on NSD (non-single diffractive) data at $\sqrt{s} = 540$ GeV became available from the CERN S$\bar{\text{p}}$pS collider (UA5 collaboration [7]), the TG failed to provide good fits. Comparison of the UA5 results with ISR data revealed a violation of KNO scaling favoring higher scaled multiplicities at 540 GeV. The effect could not be explained by any plausible admixture of single and double diffractive events in the "NSD" sample [8]. Scaling violations were further observed for charged multiplicities in different pseudorapidity intervals [9]. Finally, a comprehensive analysis of the data by the UA5 collaboration led to [10] "a new empirical regularity for multiplicity distributions in place of KNO scaling".

The UA5 analysis involved the use of the negative binomial (NB) distribution

$$P_n^{\text{NB}} = \binom{n+k-1}{n} \left(\frac{\langle n \rangle/k}{1+\langle n \rangle/k} \right)^n \left(\frac{1}{1+\langle n \rangle/k} \right)^k ,$$

(2)

where

$$\binom{n+k-1}{n} = \frac{(n+k-1)!}{n!(k-1)!} .$$

Excellent fits were obtained in which the parameters $\langle n \rangle$ and k were found to vary smoothly with the size of the pseudorapidity range for data at a given energy [9]. For full phase space NSD data of pp and $\bar{\text{p}}$p collisions in the range $11 < \sqrt{s} < 540$ GeV, it was found [10] that k^{-1} varies linearly with $\ln \sqrt{s}$. The most recent fits, which include data [11] at $\sqrt{s} = 200$ and 900 GeV, yield the parameterization

$$\langle n \rangle = A + B \ln s + C(\ln s)^2 ,$$

(3)

$$k^{-1} = a + b \ln \sqrt{s} ,$$

(4)

where $A = 2.7 \pm 0.7$, $B = -0.03 \pm 0.21$, $C = 0.167 \pm 0.016$, $a = -0.104 \pm 0.004$ and $b = 0.058 \pm 0.001$.

The NB scaled multiplicity moments, $C_q \langle n^q \rangle / \langle n \rangle^q$, are polynomials in the variables $1/\langle n \rangle$ and $1/k$. The above results show that KNO scaling, which requires energy independent moments, is not observed. The fact that in the ISR energy range the ratio $(D/\langle n \rangle)^2 = C_2 - 1 = 1/\langle n \rangle + 1/k$ is approximately constant is not due to KNO scaling in the distribution, but appears to be simple an accidental local effect resulting from the interplay of a decreasing $\langle n \rangle^{-1}$ and an increasing k^{-1} with energy.

At high energies, where $\langle n \rangle$ becomes much larger than k, the moments become functions of k only and the NB goes into the gamma distribution:

$$\langle n \rangle P_n^G = \frac{k^k}{\Gamma(k)} z^{k-1} e^{-kz} .$$

(5)

It is clear that asymptotic KNO scaling requires a constant k^{-1}. Instead, k^{-1} increases linearly with $\ln \sqrt{s}$ and there is no sign of an approach to scaling up to $\sqrt{s}$ of 900 GeV. This is not surprising, since Feynman scaling, which provides the foundation for KNO scaling, is also violated. The question really was never [8] why KNO scaling was violated at high energies, but why it appeared to hold through the entire energy range $2 < M(\text{or } \sqrt{s}) < 60$ GeV in spite of a substantial Feynman scaling violation manifested through the rise with $\ln \sqrt{s}$ of the rapidity plateau. As we have seen, the answer to this question was given by the results of the fits with the NB distribution, which show clearly that the approximate scaling observed in this energy region is simply accidental. The scaling parameter k^{-1} increases logarithmically with energy, violating KNO scaling in exactly the same

manner as Feynman scaling is violated by the logarithmic rise with energy of the rapidity plateau.

2. A new statistical description of multiplicity distributions: the modified gamma distribution. Although the NB provides excellent fits to charged multiplicity distributions, correlations between charged and neutral multiplicities do exist and therefore a proper statistical model of particle production must describe both distributions simultaneously. In addition, it must take into account charge conservation. A simple way to incorporate these requirements into a model is to assume that an interaction produces n^* fictitious intermediate neutral objects, each one of which either "decays" into a charged hadron *pair* or becomes a single neutral hadron with probability, respectively, q and $1-q$. The probability q can be evaluated at each energy by setting the ratio of the average charged to neutral multiplicity equal to $2q/(1-q)$. This ratio is approximately 2, yielding $q=1/2$. The average value of n^* is clearly $\langle n^* \rangle = \langle n_{\text{total}} \rangle/(1+q)$.

We now propose that the distribution of the n^* intermediate objects is a gamma distribution (eq. (5)), i.e.

$$\langle n^* \rangle P_{n^*}^{\mathrm{G}} = \frac{\mu^\mu}{\Gamma(\mu)} z^{\mu-1} \mathrm{e}^{-\mu z} ;$$

$$z=n^*/\langle n^* \rangle, \quad \mu^{-1}=(D^*/\langle n^* \rangle)^2 . \qquad (6)$$

The actual charged particle multiplicity distribution is then given by a "modified gamma"(MG) distribution, which takes into account the "decay" probability of the n^* into charged hadron pairs. The MG probability for $2n$ ($n=0, 1, 2, ...$) charged particles is

$$P_{2n}^{\mathrm{MG}} = \sum_{n^*=n}^\infty P_{n^*}^{\mathrm{G}} \binom{n^*}{n} q^n(1-q)^{n^*-n} . \qquad (7)$$

In addition to q, which as we discussed previously can be obtained experimentally and will be set equal to 1/2 throughout all our fits, the parameters of the MG are those of the gamma distribution, $\langle n^* \rangle$ and μ^{-1}. For $q=1/2$, $\langle n^* \rangle = \langle n \rangle$. A constant μ^{-1} is required for KNO scaling in the n^* distribution. In fitting the data, it will turn out that μ^{-1} increases logarithmically with energy in violation of KNO scal-
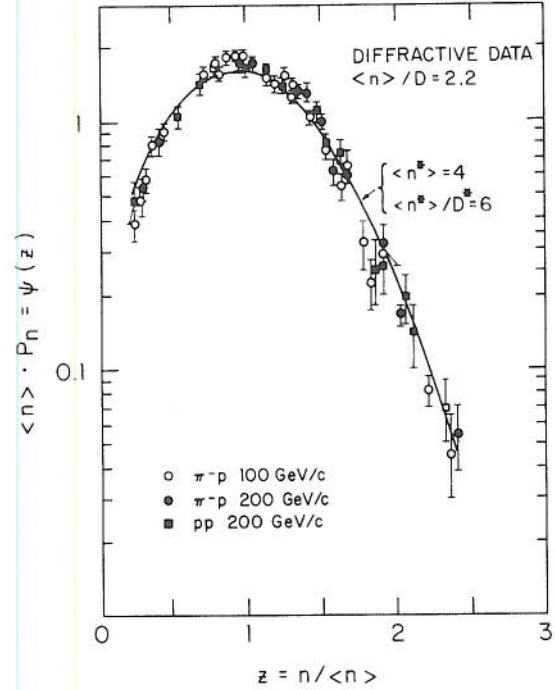

Fig. 1. The diffractive data of ref. [3] fitted with the modified gamma function.

ing. However, due to a compensating broadening of the distributions at low energies caused by the statistics of the "decay" of the n^* objects, the $D/\langle n \rangle$ ratios of the actual charged multiplicity distributions will turn out to be approximately constant for $\sqrt{s} < 60$ GeV, giving the appearance of KNO scaling in this energy region, in agreement with the data.

The MG has been used successfully to fit a variety of data obtained in hadron diffraction dissociation, in pp ($\bar{\mathrm{p}}\mathrm{p}$) collisions and $\mathrm{e}^+\mathrm{e}^-$ annihilation. The results, which are presented below, lead to a new concept of universality for multiplicity distributions.

3. Hadronic multiplicities. Examples of fits to full phase space hadronic data using the MG distribution are shown in figs. 1 and 2. Fig. 1 represents a fit to single diffractive data in the (available) mass range $1 < M < 6$ GeV obtained in proton and pion hadron dissociation on hydrogen [3]. Since these data are not available in table form, the fit was made "by eye". Considering the disparity of the data (various diffractive masses and two types of dissociating hadrons), the fit is quite satisfactory. Although $\langle n \rangle/D=2.2$ for the data, a value of $\langle n^* \rangle/D^*=6$ was used to obtain the curve shown in fig. 1. The broad-
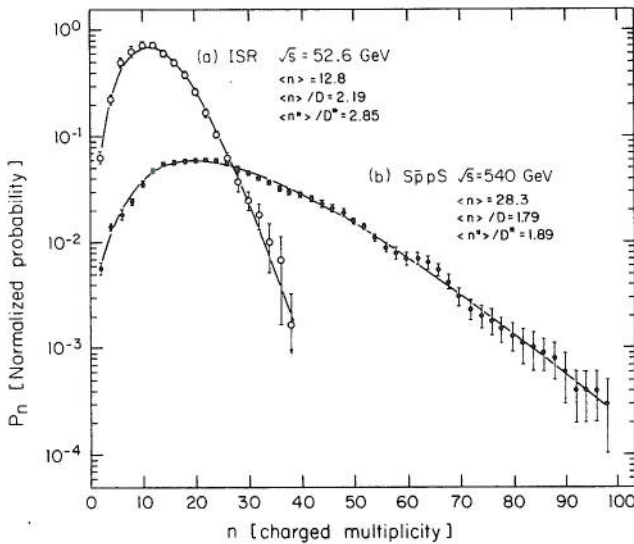
Fig. 2. Full phase space inelastic non-single-diffractive data fitted with the modified gamma function: (a) ISR data [5] at $\sqrt{s}=52.6$ GeV and (b) collider data [7] at $\sqrt{s}=540$ GeV.

ening of the distribution caused by the statistics in the "decay" of the n^* objects is substantial in this low-energy region. In fig. 2a we present a fit to NSD data [5] obtained at $\sqrt{s}=52.6$ GeV at the ISR. This fit, which has $\chi^2=0.5$/d.o.f., yields $\langle n^*\rangle/D^*=2.85\pm0.04$. The value of $\langle n\rangle/D$ for the data in this case is 2.19 ± 0.05, very close to that of the diffractive data. Thus, as mentioned previously, while $\langle n^*\rangle/D^*$ decreases (the distribution broadens) with energy in violation of KNO scaling, the $\langle n\rangle/D$ of the data remains approximately constant, giving the appearance of scaling. Finally, fig. 2b shows a fit to collider NSD data [7] at $\sqrt{s}=540$ GeV. This fit has $\chi^2=0.8$/d.o.f. and yields $\langle n^*\rangle/D^*=1.89\pm0.02$, a value close to that of the data, $\langle n\rangle/D=1.79\pm0.02\pm0.06$. One sees that as $\langle n\rangle$ increases, the actual charged-particle multiplicity distribution deviates less and less from the n^* distribution.

The values of the parameter $\mu=(\langle n^*\rangle/D^*)^2$ of the MG fits to the ISR and collider data are, respectively, 8.13 ± 0.23 and 3.57 ± 0.04. These values are very close to the values of the parameter k of the NB fits [10] to the same data, namely $k=7.9\pm0.3$ and 3.69 ± 0.09 (see also eq. (4)). Recalling that for the negative binomial distribution $(D/\langle n\rangle)^2=k^{-1}+\langle n\rangle^{-1}$, the fact that μ turns out to be numerically the same as k means that $(D/\langle n\rangle)^2=(D^*/\langle n^*\rangle)^2+\langle n\rangle^{-1}$, i.e. the scaled variance of the

actual charged multiplicity distribution is larger than that of the n^* distribution by the value of $\langle n\rangle^{-1}$. This is exactly the result expected (for large $\langle n\rangle$) from the statistical superposition of the primary gamma distribution of the n^* and their binomial "decay" into charged pairs with probability $q=1/2$. The effect of the binomial "decay" is to modify the gamma distribution into an apparent negative binomial.

As mentioned previously, $q=\frac{1}{2}$ corresponds to a ratio of charged to neutral particles $R=\langle n\rangle/\langle n_0\rangle=2$. The effect of a different q value would be to replace the $\langle n\rangle^{-1}$ term in the scaled variance given above by the factor $2(1-q)\langle n\rangle^{-1}$. For example, for $q=\frac{1}{3}$, corresponding to $R=1$ (a value closer to reality at high energies), the term $\langle n\rangle^{-1}$ would be replaced by $\frac{4}{3}\langle n\rangle^{-1}$. This change would have a very small effect on the distribution, since at these energies the scaled variance is dominated by the μ^{-1} term. At lower energies, where μ^{-1} is small and the effect could have been appreciable, the ratio R is very close to 2 and hence $q=\frac{1}{2}$ is the correct value to use. Using $q=\frac{1}{2}$ throughout has negligible effect on the fits.

We now turn to a more detailed comparison between the parameters μ of the MG of k of the NB distribution. At energies below the ISR range, where $\langle n\rangle<k$, the two parameters are no longer numerically the same. For example, a MG fit to fixed target pp ND data [12] at $\sqrt{s}=13.8$ GeV (for which $\langle n\rangle=7.3$) yields $\mu=15.4\pm0.3$ as compared to $k=21\pm2$ of the NB fit [10]. A parameterization which preserves the equality of μ and k at high energies (where $\langle n\rangle>k$) and also gives the correct μ values at low energies (where $\langle n\rangle<k$) is given by

$$\mu^{-1}=a+b\ln(M+c),\qquad(8)$$

where M is the "available" mass in GeV, which for nucleon–nucleon collisions is equal to $\sqrt{s}-1.88$ GeV. With a and b numerically the same as in the expression for k^{-1} (eq. (4)) and a value of $c=6$ GeV, eq. (8) gives correct μ values for all the data discussed, including the diffractive data for which $M\sim4$ GeV. The values of $\langle n^*\rangle/D^*=\sqrt{\mu}$ obtained at $M=4$, $\sqrt{s}=13.8$, 52.6 and 540 GeV are, respectively, 5.8, 4.0, 2.8 and 1.95, as compared to the MG-fit values of 6, 3.9 ± 0.1, 2.85 ± 0.04 and 1.89 ± 0.02. We finally note that, with the chosen values of a, b and c in eq.
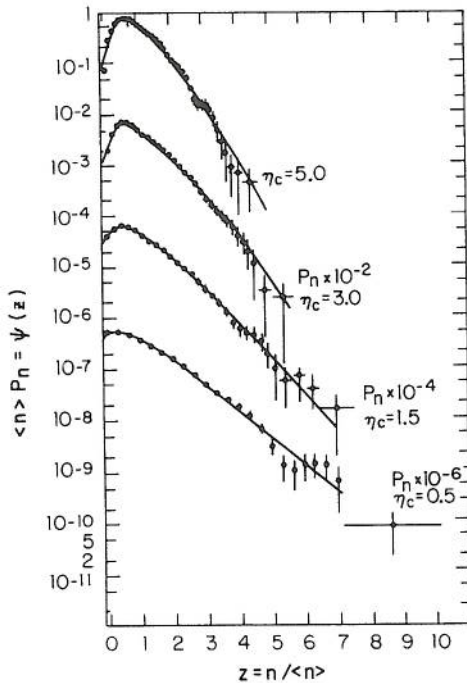
Fig. 3. Fits with the modified gamma function of charged-particle multiplicity distributions in restricted pseudorapidity intervals, defined by $|\eta| < \eta_c$, at $\sqrt{s} = 540$ GeV (data from ref. [9]).

(8), μ is positive all the way down to $M=0$, as it should be.

In summary, the hadronic charged-particle multiplicity distribution at a given energy may be obtained from eq. (7) using μ^{-1} from eq. (8) and $\langle n^* \rangle = \langle n \rangle / 2q$, with $\langle n \rangle$ taken from eq. (3) and q evaluated from the ratio of the charged to neutral average multiplicities, $\langle n \rangle / \langle n_0 \rangle = 2q/(1-q)$. This procedure fits well both diffractive and NSD inclusive distributions, preserving the universality concept of ref. [4] discussed in the introduction. It also fits well distributions limited in pseudorapidity space, as shown in the examples of fig. 3. The curves in this figure were obtained by using as parameters $\langle n^* \rangle$ and μ^{-1} in the MG distribution for each pseudorapidity interval the values $\langle n \rangle$ and k^{-1} of the NB fits of ref. [9].

4. e^+e^- *multiplicities*. The e^+e^- charged multiplicity distributions differ from the hadronic ones in two ways: they have a more symmetrical shape and they are narrower. This is illustrated in fig. 4a, in which e^+e^- data [13], plotted in KNO form, are compared with a curve (broken line) representing
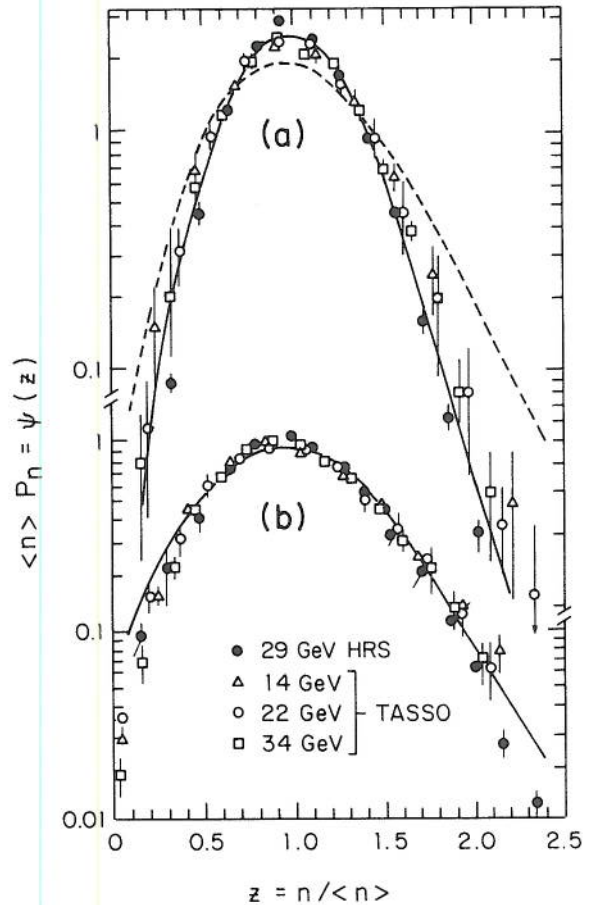

Fig. 4. Comparison of e^+e^- charged multiplicity distributions [13], plotted in KNO form, with curves generated using the modified gamma procedure discussed in the text. (a) Inclusive data compared with a non-diffractive hadronic distribution at the same range of $\langle n \rangle$ values (broken line) and with a distribution (solid line) representing the statistical superposition of two independent single-jet distributions (given by the line in (b)). (b) Single-jet data compared with the ND hadronic distribution expected at the same $\langle n \rangle$ value as that of the jet.

the hadronic distribution for the same range of $\langle n \rangle$ values. The widt of e^+e^- distribution is smaller than the hadronic width by about a factor of $\sqrt{2}$. Realizing that in e^+e^- annihilation the final state consists mostly of two jets, single-jet distributions were also measured. It was found that the distributions of the two jets are almost entirely uncorrelated and that the width of each distribution is approximately the same as that of a ND hadronic distribution of the same $\langle n \rangle$ value as that of the jet, which is one half of the $\langle n \rangle$ value of the inclusive e^+e^- sample. The statistical addition of these two independent jet dis-

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tributions results, of course, in a distribution with an $\langle n \rangle / D$ ratio smaller than that of the single-jet sample by a factor of $\sqrt{2}$.

Using our MG description of hadronic charged multiplicities, we have carried these findings one step further by comparing not only widths but entire single-jet distributions with hadronic ones. The comparison is best done in KNO form, so that data at different $\langle n \rangle$ values can be superimposed. The result is shown in fig. 4b, in which the line represents an MG distribution obtained by using $\langle n \rangle = 6$ and $\langle n^* \rangle / D^* = 4.5$. In view of the expected experimental biases at the edges of the distribution introduced by the inevitable "cross-talk" between the two jets, the agreement is quite satisfactory. Thus, in addition to single diffractive and NSD hadronic data, single-jet e^+e^- data are also described well by the MG distribution. The statistical addition of two independent single-jet MG distributions is represented by the solid line in fig. 4a. The excellent agreement between data and curve in this case, where the abovementioned cross-talk effects are absent, strengthens the conclusion that single-jet and hadronic distributions are identical.

5. Universality of multiplicity distributions. We have been above that diffractive, ND hadronic and single-jet e^+e^- multiplicity distributions are described well by the same parameterization of the MG distribution. Since a different number of quarks is involved in each case, it appears that the hadronization process is not affected by the initial conditions. This concept of universality has been discussed previously in connection with diffractive and ND distributions [4]. The new element introduced in this paper is the extension of the universality concept to conclude e^+e^- single-jet distributions.

6. Conclusions. The modified gamma distribution describes well both hadronic and e^+e^- multiplicities arising form a clearly identifiable piece of hadronic matter, i.e. a diffractive mass, a single-jet or an inclusive non-diffractive hadronic collision. All such multiplicities can be generated using the same parameterization in the MG distribution, a fact that leads to a new concept in universality. The MG description takes into account charge conservation and provides neutral multiplicities simultaneously with the charged, on an event-by-event basis. The simplicity of the underlying gamma distribution and the expanded concept of universality may provide new theoretical insight into the process of hadronization.

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